Solving two-mode squeezed harmonic oscillator and kth-order harmonic generation in Bargmann-Hilbert spaces

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Abstract

We analyze the two-mode squeezed harmonic oscillator and the kth-order harmonic generation within the framework of Bargmann-Hilbert spaces of entire functions. For the displaced, single-mode squeezed and two-mode squeezed harmonic oscillators, we derive the exact, closed-form expressions for their energies and wave functions. For the kth-order harmonic generation with $k \geq 3$, our result indicates that it does not have eigenfunctions and is thus ill-defined in the Bargmann-Hilbert space.

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1 Introduction

Recently there is renewed interest in formulating and solving dynamical systems involving harmonic modes in the framework of Bargmann-Hilbert spaces [1, 2, 3]. For example, in [4, 5], we applied the Bargmann-Hilbert space technique to obtain the exact solutions of families of quantum non-linear optical as well as spin-boson models. In [6, 7, 8, 9, 10], the authors applied the technique to the quantum Rabi model, a simple system describing the interaction of a two-level atom with a harmonic mode. A Bargmann-Hilbert space is a Hilbert space of entire functions introduced by Bargmann and Segal. It is a vector space with typical orthonormal basis $\frac{z^n}{\sqrt{n!}}, n = 0, 1, 2, \cdots$. Elements in the space are entire functions and the space is equipped with a well-defined Hermitian scalar product,

$$(f, h) = \int \overline{f(z)}\, h(z)\, d\mu(z) \tag{1.1}$$

for any two elements $f(z), h(z)$ in the space, where $d\mu(z) = \frac{1}{\pi} e^{-|z|^2}\, dx\, dy$. In a Bargmann-Hilbert space, the harmonic creation and annihilation operators $a^\dagger, a$ can be realized as

$a^\dagger \to z,\ a \to \frac{d}{dz}$. This realization enables one to convert the time-independent Schrödinger equation of a dynamical system into a differential equation. Solutions to the differential equation are entire functions.

In this paper, we apply Bargmann-Hilbert spaces to analyze the two-mode squeezed harmonic oscillator and kth order harmonic generation. For the cases of the displaced, single-mode squeezed and two-mode squeezed harmonic oscillators, we derive the exact, closed-form expressions for their energies and wave functions. For the kth-order harmonic generation with $k \geq 3$, our result shows that it does not have eigenfunctions and thus is ill-defined in the Bargmann-Hilbert space.

2 Two-mode squeezed harmonic oscillator

The Hamiltonian of the two-mode squeezed harmonic oscillator reads

$$H = \omega(a_1^\dagger a_1 + a_2^\dagger a_2) + g\,(a_1^\dagger a_2^\dagger + a_1 a_2), \tag{2.1}$$

where we assume that the boson modes are degenerate with the same frequency ω, and g is a real constant. In terms of the operators $K_\pm, K_0$ defined by

$$K_+ = a_1^\dagger a_2^\dagger, \qquad K_- = a_1 a_2, \qquad K_0 = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1), \tag{2.2}$$

the Hamiltonian (2.1) can be written as

$$H = 2\omega\left(K_0 - \frac{1}{2}\right) + g\,(K_+ + K_-). \tag{2.3}$$

The operators $K_\pm, K_0$ form the $su(1,1)$ Lie algebra. The quadratic Casimir of the algebra, $C = K_+K_- - K_0(K_0-1)$, takes the particular values $C = \kappa(1-\kappa)$ in the representation (2.2), where $\kappa = 1/2, 1, 3/2, \cdots$. This is the well-known infinite-dimensional unitary irreducible representation of $su(1,1)$ known as the positive discrete series $\mathcal{D}^+(\kappa)$. Thus the Fock-Hilbert space decomposes into the direct sum of infinite subspaces $\mathcal{H}^\kappa$ labeled by $\kappa = 1/2, 1, 3/2, \cdots$.

The basis state in the subspace $\mathcal{H}^\kappa$, denoted as $|\kappa, n\rangle,\ n = 0, 1, 2, \cdots$, has the form

$$|\kappa, n\rangle = \frac{(a_1^\dagger)^{n+2\kappa-1}(a_2^\dagger)^n}{\sqrt{(n+2\kappa-1)!n!}}|0\rangle, \tag{2.4}$$

and the action of $K_\pm, K_0$ in this representation is given by

$$\begin{aligned} K_0|\kappa,n\rangle &= (n+\kappa)|\kappa,n\rangle, \\ K_+|\kappa,n\rangle &= \sqrt{(n+2\kappa)(n+1)}|\kappa,n+1\rangle, \\ K_-|\kappa,n\rangle &= \sqrt{(n+2\kappa-1)n}|\kappa,n-1\rangle. \end{aligned} \tag{2.5}$$

Using the Fock-Bargmann correspondence

$$a^\dagger \longrightarrow z, \qquad a \longrightarrow \frac{d}{dz}, \qquad |n\rangle \longrightarrow \frac{z^n}{\sqrt{n!}}, \tag{2.6}$$

we can show that the infinite set of monomials

$$\Psi_{\kappa,n}(z) = \frac{z^n}{\sqrt{(n+2\kappa-1)!n!}}, \qquad n = 0,1,2,\cdots, \tag{2.7}$$

form the basis in the Bargmann-Hilbert subspace associated with the representation (2.5). Thus the operators $K_\pm, K_0$ (2.2) have the single-variable differential realization in the subspace labeled by the Bargmann index κ,

$$K_0 = z\frac{d}{dz} + \kappa, \quad K_+ = z, \quad K_- = z\frac{d^2}{dz^2} + 2\kappa\frac{d}{dz}, \quad \kappa = 1/2, 1, 3/2, \cdots. \tag{2.8}$$

By means of this differential representation (2.8), we can express the Hamiltonian (2.3) (i.e. (2.1)) as the 2nd-order differential operator in each Bargmann-Hilbert subspace labeled by κ,

$$H = 2\omega\left(z\frac{d}{dz} + \kappa - \frac{1}{2}\right) + g\left(z + z\frac{d^2}{dz^2} + 2\kappa\frac{d}{dz}\right). \tag{2.9}$$

Then the time-independent Schrödinger equation gives the differential equation for wave-function $\psi(z)$,

$$gz\frac{d^2}{dz^2}\psi(z) + 2(\omega z + g\kappa)\frac{d}{dz}\psi(z) + \left[gz + 2\omega\left(\kappa - \frac{1}{2}\right) - E\right]\psi(z) = 0. \tag{2.10}$$

With the substitution

$$\psi(z) = e^{-\frac{\omega}{g}(1-\Lambda)z}\varphi(z), \qquad \Lambda = \sqrt{1 - \frac{g^2}{\omega^2}}, \tag{2.11}$$

where $\left|\frac{g}{\omega}\right| < 1$, it follows,

$$\mathcal{L}\varphi \equiv \left\{gz\frac{d^2}{dz^2} + 2[\omega\Lambda z + g\kappa]\frac{d}{dz} + 2\kappa\omega\Lambda - \omega - E\right\}\varphi = 0. \tag{2.12}$$

This differential equation is exactly solvable, This is seen as follows. First of all, let us recall the characterization of exact solvability of a differential operator. A linear differential operator $\mathcal{L}$ is exactly solvable if it preserves an infinite flag of finite-dimensional functional spaces,

$$\mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_\mathcal{M} \subset \cdots,$$

whose bases admit explicit analytic forms, that is there exists a sequence of finite-dimensional invariant subspaces $\mathcal{V}_\mathcal{M}$, $\mathcal{M} = 1, 2, 3, \cdots$,

$$\mathcal{L}\mathcal{V}_\mathcal{M} \subset \mathcal{V}_\mathcal{M}, \quad \dim\mathcal{V}_\mathcal{M} < \infty, \quad \mathcal{V}_\mathcal{M} = \mathrm{span}\{\xi_1, \cdots, \xi_{\dim\mathcal{V}_\mathcal{M}}\}.$$

In our case, we have, for any positive integer n,

$$\mathcal{L}z^n = [(2n+2\kappa)\omega\Lambda - \omega - E]\, z^n + n(n+2\kappa-1)gz^{n-1}. \tag{2.13}$$

It follows that $\mathcal{L}$ preserves an infinite flag of finite dimensional spaces $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_{\mathcal{M}} \subset \cdots$, with explicitly determined subspaces $\mathcal{V}_{\mathcal{M}} = \{1, z, z^2, \cdots, z^{\mathcal{M}}\}$, and exact solutions are polynomials in z in the Bargmann-Hilbert space. We thus seek solutions of the form to the differential equation (2.12),

$$\varphi(z) \;=\; \prod_{i=1}^{\mathcal{M}}(z-z_i), \qquad \mathcal{M} = 0, 1, 2, \cdots, \tag{2.14}$$

where $\varphi(z) \equiv 1$ for $\mathcal{M} = 0$, $\mathcal{M}$ is the degree of the polynomial and z_i are roots of the polynomial to be determined. Substituting into (2.12) and dividing on both sides by $\varphi(z)$ give rise to

$$\begin{aligned} E + \omega - 2\kappa\omega\Lambda &= gz\sum_{i=1}^{\mathcal{M}}\frac{1}{z-z_i}\sum_{j\neq i}^{\mathcal{M}}\frac{2}{z_i-z_j} + 2[\omega\Lambda z + g\kappa]\sum_{i=1}^{\mathcal{M}}\frac{1}{z-z_i} \\ &= 2n\omega\Lambda + \sum_{i=1}^{\mathcal{M}}\frac{\mathrm{Res}_{z=z_i}}{z-z_i}, \end{aligned} \tag{2.15}$$

where $\mathrm{Res}_{z=z_i}$ are the residues of the right hand side of the first equality at the simple poles $z = z_i$, i.e.

$$\mathrm{Res}_{z=z_i} = gz_i\sum_{j\neq i}^{\mathcal{M}}\frac{2}{z_i-z_j} + 2\omega\Lambda z_i + 2g\kappa. \tag{2.16}$$

The left hand side (2.15) is a constant and the right hand side is a meromorphic function with simples poles at $z = z_i$. The right hand side is a constant if and only if the coefficient of all the residues at the simple poles are vanishing. We thus obtain the energies

$$E = -\omega + \left[2\mathcal{M} + 2\left(\kappa - \frac{1}{2}\right) + 1\right]\omega\Lambda, \tag{2.17}$$

and the system of algebraic equations satisfied by the roots z_i,

$$\sum_{j\neq i}^{\mathcal{M}}\frac{1}{z_i-z_j} + \frac{\omega}{g}\Lambda + \frac{\kappa}{z_i} = 0, \qquad i = 1, 2, \cdots, \mathcal{M}. \tag{2.18}$$

The corresponding wavefunctions are given by

$$\psi(z) = e^{-\frac{\omega}{g}(1-\Lambda)z}\prod_{i=1}^{\mathcal{M}}(z-z_i). \tag{2.19}$$

3 kth-order harmonic generation

The Hamiltonian of the kth-order harmonic generation reads

$$H = \omega a^\dagger a + g\left[(a^\dagger)^k + a^k\right], \tag{3.1}$$

where $k = 1, 2, \cdots$ is any positive integer, and g is a real constant. The $k = 1$ and $k = 2$ cases of (3.1) give the Hamiltonians of the displaced and single-mode squeezed harmonic oscillators, respectively These two oscillator models can be solved by the single-mode Bogoliubov transformation [11]. For $k \geq 3$, (3.1) gives models with higher order harmonic generation.

Introduce the operators $Q_\pm, Q_0$ in terms of the harmonic mode,

$$Q_+ = \frac{1}{(\sqrt{k})^k}(a^\dagger)^k, \qquad Q_- = \frac{1}{(\sqrt{k})^k}a^k, \qquad Q_0 = \frac{1}{k}\left(a^\dagger a + \frac{1}{k}\right). \tag{3.2}$$

Then in terms of $Q_\pm, Q_0$, the Hamiltonian (3.1) can be written as

$$H = k\omega\left(Q_0 - \frac{1}{k^2}\right) + g\sqrt{k^k}\,\left(Q_+ + Q_-\right). \tag{3.3}$$

It can be shown [4] that the operators $Q_\pm, Q_0$ form a polynomial algebra of degree $k-1$, defined by the commutation relations

$$\begin{aligned} [Q_0, Q_\pm] &= \pm Q_\pm, \\ [Q_+, Q_-] &= \phi^{(k)}(Q_0) - \phi^{(k)}(Q_0 - 1), \end{aligned} \tag{3.4}$$

where

$$\phi^{(k)}(Q_0) = -\prod_{i=1}^{k}\left(Q_0 + \frac{i}{k} - \frac{1}{k^2}\right) + \prod_{i=1}^{k}\left(\frac{i-k}{k} - \frac{1}{k^2}\right) \tag{3.5}$$

is a k^{th}-order polynomial in Q_0. The Casimir operator of the algebra is given by

$$C = Q_- Q_+ + \phi^{(k)}(Q_0) = Q_+ Q_- + \phi^{(k)}(Q_0 - 1). \tag{3.6}$$

For $k = 1$ and $k = 2$, (3.4) reduces to the Heisenberg and $su(1,1)$ algebras, respectively. Thus, the algebra (3.4) can be viewed as polynomial deformation of $su(1,1)$ and Heisenberg Lie algebras.

The realization (3.2) provides a unitary irreducible representation of the polynomial algebra, which for $k = 2$ reduces to the well-known positive discrete series of $su(1,1)$. In the realization, the Casimir (3.6) takes the particular value,

$$C = \prod_{i=1}^{k}\left(\frac{i-k}{k} - \frac{1}{k^2}\right). \tag{3.7}$$

If we use q to label the basis states of this representation in the Fock-Hilbert space $\mathcal{H}_b$, then it can be shown that q takes k values,

$$q = \frac{1}{k^2}, \; \frac{k+1}{k^2}, \; \frac{2k+1}{k^2}, \cdots, \; \frac{(k-1)k+1}{k^2}. \tag{3.8}$$

For $k = 2$, then $C = \frac{3}{16}$ and q equals to $\frac{1}{4}, \frac{3}{4}$, as expected. Thus the boson realization (3.2) corresponds to the infinite dimensional unitary representation with particular q values

(3.8), and the Fock space $\mathcal{H}_b$ decomposes into the direct sum $\mathcal{H}_b = \mathcal{H}_b^{\frac{1}{k^2}} \oplus \cdots \oplus \mathcal{H}_b^{\frac{(k-1)k+1}{k^2}}$ of k irreducible components $\mathcal{H}_b^{\frac{1}{k^2}}, ..., \mathcal{H}_b^{\frac{(k-1)k+1}{k^2}}$.

The basis state $|q,n\rangle,\ n = 0, 1, \cdots$, in the irreducible representation space $\mathcal{H}_b^q$ is then given by [4]

$$|q,n\rangle = \frac{a^{\dagger k(n+q-\frac{1}{k^2})}}{\sqrt{\left[k\left(n+q-\frac{1}{k^2}\right)\right]!}}|0\rangle, \qquad n = 0, 1, 2, \cdots. \tag{3.9}$$

The action $Q_0, Q_\pm$ in this representation reads

$$\begin{aligned} Q_0|q,n\rangle &= (q+n)|q,n\rangle, \\ Q_+|q,n\rangle &= \prod_{i=1}^{k}\left(n+q+\frac{ik-1}{k^2}\right)^{\frac{1}{2}}|q,n+1\rangle, \\ Q_-|q,n\rangle &= \prod_{i=1}^{k}\left(n+q-\frac{(i-1)k+1}{k^2}\right)^{\frac{1}{2}}|q,n-1\rangle. \end{aligned} \tag{3.10}$$

By using the Fock-Bargmann correspondence (2.6), we can make the following association

$$|q,n\rangle \longrightarrow \Psi_{q,n}(z) = \frac{z^n}{\sqrt{\left(k(n+q-\frac{1}{k^2})\right)!}}. \tag{3.11}$$

It can then be shown that in each subspace $\mathcal{H}^q$ of the Bargmann-Hilbert space with basis vectors $\Psi_{q,n}(z)$, the operators $Q_\pm, Q_0$ (3.2) are realized by single-variable differential operators

$$\begin{aligned} Q_0 &= z\frac{d}{dz}+q, \\ Q_+ &= \frac{z}{(\sqrt{k})^k}, \\ Q_- &= z^{-1}(\sqrt{k})^k\prod_{j=1}^{k}\left(z\frac{d}{dz}+q-\frac{(j-1)k+1}{k^2}\right). \end{aligned} \tag{3.12}$$

We remark that there is no singularity in the differential operator expression for Q_-. This is because the z^{-1} term disappears in the expansion of the product in Q_- thanks to the fact that $\prod_{j=1}^{k}\left(q-\frac{(j-1)k+1}{k^2}\right) \equiv 0$ for all the allowed q values.

Using the differential realization (3.12) we can equivalently write (3.3) (i.e. (3.1)) as the kth-order single-variable differential operator in each Bargmann-Hilbert subspace labelled by index q,

$$H = k\omega\left(z\frac{d}{dz}+q-\frac{1}{k^2}\right)+g\left[z+k^k\,z^{-1}\prod_{j=1}^{k}\left(z\frac{d}{dz}+q-\frac{(j-1)k+1}{k^2}\right)\right]. \tag{3.13}$$

Thus the time-independent Schrödinger equation for the model yields

$$\left\{gk^k\,z^{-1}\prod_{j=1}^{k}\left(z\frac{d}{dz}+q-\frac{(j-1)k+1}{k^2}\right)+gz+k\omega\left(z\frac{d}{dz}+q-\frac{1}{k^2}\right)-E\right\}\psi = 0. \tag{3.14}$$

This is a k-th order differential equation of Fuchs' type. Solutions to this equation must be analytic in the whole complex plane if E belongs to the spectrum of H. In other words, we are seeking solution of the form

$$\psi(z) = \sum_{n=0}^{\infty} K_n(E)\, z^n, \tag{3.15}$$

which converges in the entire complex plane, i.e. solution $\psi(z)$ which is entire.

Substituting (3.15) into (3.14), we obtain the 3-step recurrence relation,

$$\begin{aligned} &K_1(E) + A_0\, K_0(E) = 0, \\ &K_{n+1}(E) + A_n\, K_n(E) + B_n\, K_{n-1}(E) = 0, \qquad n \geq 1, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} A_n &= \frac{\omega\left(n + q - \frac{1}{k^2} - \frac{E}{k\omega}\right)}{g\, k^{k-1} \prod_{j=1}^{k}\left(n + 1 + q - \frac{(j-1)k+1}{k^2}\right)}, \\ B_n &= \frac{1}{k^k \prod_{j=1}^{k}\left(n + 1 + q - \frac{(j-1)k+1}{k^2}\right)}. \end{aligned} \tag{3.17}$$

The coefficients A_n, B_n have the behavior when $n \to \infty$,

$$A_n \sim a\, n^{\alpha}, \qquad B_n \sim b\, n^{\beta} \tag{3.18}$$

with

$$a = \frac{\omega}{g k^{k-1}}, \qquad \alpha = -k + 1, \qquad b = \frac{1}{k^k}, \qquad \beta = -k. \tag{3.19}$$

Thus the asymptotic structure of solutions to the $n \geq 1$ part of (3.16) depends on the Newton-Puiseux diagram formed with the points $P_0(0,0), P_1(1, -k+1), P_2(2, -k)$ [12]. Let σ be the slope of $\overline{P_0 P_1}$ and τ the slope of $\overline{P_1 P_2}$ so that $\sigma = \alpha$ and $\tau = \beta - \alpha$. Applying the Perron-Kreuser theorem (i.e. Theorem 2.3 of [12]), we have

<u>$k = 1$ **case: the displaced harmonic oscillator**</u>. $\sigma = 0, \tau = -1$, that is $\sigma > \tau$. In this case, the truly 3-term part (i.e. the $n \geq 1$ part) of the recurrence relation (3.16) has two linearly independent solutions $K_{n,1}, K_{n,2}$ for which, when $n \to \infty$

$$\frac{K_{n+1,1}}{K_{n,1}} \sim -\frac{\omega}{g k^{k-1}}, \qquad \frac{K_{n+1,2}}{K_{n,2}} \sim -\frac{g}{k\omega}\, n^{-1} \tag{3.20}$$

So $K_n^{min} \equiv K_{n,2}$ is a minimal solution of the truly 3-term part of (3.16) with $k = 1$. The corresponding infinite power series solution is generated by substituting K_n^{min} for the K_n's in (3.15) and converges in the whole complex plane, i.e. it is entire.

<u>$k = 2$ **case: the squeezed harmonic oscillator**</u>. $\sigma = -1, \tau = -1$, and so $\sigma = \tau = \alpha$. The characteristic equation of the $n \geq 1$ part of (3.16), $t^2 = 0$, has two equal solutions $t_1 = t_2 = 0$. Then all solutions of (3.16) behave similarly as $n \to \infty$, viz,

$$\lim_{n \to \infty} \sup\, (|K_n|\, n!)^{\frac{1}{n}} = 0 \tag{3.21}$$

for all non-trivial solutions of the 2nd equation of (3.16) with $k = 2$. The zero limit means that the $n \geq 1$ part (i.e. the truly 3-term part) of the recurrence (3.16) possesses a minimal solution K_n^{min} and the corresponding infinite power series expansion, obtained by substituting K_n^{min} for the K_n's in (3.15), converges in the whole complex plane, i.e. it is entire.

<u>$k \geq 3$ **case**</u>: $\sigma = -k+1, \tau = -1$, and thus point P_1 lies below the line segment $\overline{P_0 P_1}$ in the Newton-Puiseux diagram. Then

$$\lim_{n\to\infty} \sup \left(|K_n|\,(n!)^{\frac{k}{2}}\right)^{\frac{1}{n}} = \frac{1}{\sqrt{k^k}} \tag{3.22}$$

for all non-trivial solutions of the 2nd equation of (3.16). This indicates that solutions to the truly 3-term part of the recurrence (3.16) with $k \geq 3$ are dominant, and the corresponding infinite series expansion (3.15) has a finite radius of convergence proportional to $\sqrt{k^k}$. It follows that there does not exist solution to (3.14) with $k \geq 3$ which is entire. We thus conclude that the kth-order harmonic generation model with $k \geq 3$ does not have eigenfunctions (and is ill-defined) in the Bargmann-Hilbert space.

By the Pincherle theorem, i.e. Theorem 1.1 of [12], the ratios of successive elements of the minimal solution sequences K_n^{min} for the $k = 1, 2$ cases are expressible in terms of infinite continued fractions. Proceeding in the direction of increasing n, we find

$$R_n = \frac{K_{n+1}^{min}}{K_n^{min}} = -\frac{B_{n+1}}{A_{n+1}-}\,\frac{B_{n+2}}{A_{n+2}-}\,\frac{B_{n+3}}{A_{n+3}-}\cdots, \tag{3.23}$$

which for $n = 0$ gives

$$R_0 = \frac{K_1^{min}}{K_0^{min}} = -\frac{B_1}{A_1-}\,\frac{B_2}{A_2-}\,\frac{B_3}{A_3-}\cdots. \tag{3.24}$$

Note that the ratio $R_0 = \frac{K_1^{min}}{K_0^{min}}$ involves K_0^{min}, although the above continued fraction expression is obtained from the truly 3-term part of (3.16), i.e the recurrence (3.16) for $n \geq 1$. However, for single-ended sequences such as those appearing in the infinite power series expansion (3.15), the ratio $R_0 = \frac{K_1^{min}}{K_0^{min}}$ of the first two terms of a minimal solution is unambiguously fixed by the $n = 0$ part (i.e. the first equation) of the recurrence (3.16), namely,

$$R_0 = -A_0 = -\frac{\omega\left(q - \frac{1}{k^2} - \frac{E}{k\omega}\right)}{g\,k^{k-1}\prod_{j=1}^{k}\left(1 + q - \frac{(j-1)k+1}{k^2}\right)}. \tag{3.25}$$

In general, the R_0 computed from (3.24) is not the same as that from (3.25) (i.e. (3.24) and (3.25) are not both satisfied) for arbitrary values of recurrence coefficients A_n and B_n. As a result, general solutions to the recurrence (3.16) are dominant and are usually generated by simple forward recursion from a given value of K_0. Physical meaningful solutions are those that are entire in the Bargmann-Hilbert spaces. They can be obtained

if E can be adjusted so that equations (3.24) and (3.25) are both satisfied. Then the resulting solution sequence $K_n(E)$ will be purely minimal and the power series expansion (3.15) will converge in the whole complex plane.

Therefore, if we define the function $F(E) = R_0 + A_0$ with R_0 given by the continued fraction in (3.24), then the zeros of $F(E)$ correspond to the points in the parameter space where the condition (3.25) is satisfied. In other words, $F(E) = 0$ yields the eigenvalue equation, which may be solved for E by standard nonlinear root-search techniques. Only for the denumerable infinite values of E which are the roots of $F(E) = 0$, do we get entire solutions of the differential equations.

As a matter of fact, the spectra for the $k = 1, 2$ cases can be determined explicitly. As will be seen in the next two subsections, the infinite power series in (3.15) actually truncates for these two cases, so that their solutions are given by polynomials in Bargmann-Hilbert spaces.

3.1 Displaced harmonic oscillator

The displaced harmonic oscillator is the $k = 1$ special case of the kth-order harmonic generation. By (3.14), the time-independent Schrödinger equation in the Bargmann-Hilbert space of analytic functions reads

$$(\omega z + g)\frac{d\psi}{dz} + (gz - E)\psi = 0. \tag{3.26}$$

With the substitution

$$\psi(z) = e^{-gz/\omega}\phi(z), \tag{3.27}$$

the above differential equation reduces to

$$\left[(\omega z + g)\frac{d}{dz} - \left(E + \frac{g^2}{\omega}\right)\right]\phi(z) = 0. \tag{3.28}$$

This differential equation is exactly solvable, and exact solutions are polynomial of the form

$$\phi(z) = \prod_{i=1}^{\mathcal{N}}(z - z_i), \qquad \mathcal{N} = 0, 1, 2, \cdots, \tag{3.29}$$

where $\phi(z) \equiv 1$ for $\mathcal{N} = 0$, $\mathcal{N}$ is the degree of the polynomial and z_i are the roots of the polynomial to be determined. Following the procedure similar to that in the last section, we obtain the energies of the system,

$$E = \omega\left(\mathcal{N} - \frac{g^2}{\omega^2}\right), \tag{3.30}$$

and the set of algebraic equations determining the roots z_i, $\omega z_i + g = 0,\ i = 1, 2, \cdots, \mathcal{N}$. It follows that $z_i = -\frac{g}{\omega}$ and the solution (3.29) has the form

$$\phi(z) = \prod_{i=1}^{\mathcal{N}}\left(z + \frac{g}{\omega}\right) = \left(z + \frac{g}{\omega}\right)^{\mathcal{N}}. \tag{3.31}$$

Thus the wavefunction of the model is given by

$$\psi(z) = e^{-\frac{g}{\omega}z}\left(z + \frac{g}{\omega}\right)^{\mathcal{N}}. \tag{3.32}$$

These expressions for the energies and wavefunction agree with those in [1] by a different approach.

3.2 Single-mode squeezed harmonic oscillator

The Hamiltonian of the single-mode squeezed harmonic oscillator is given by

$$H = \omega a^\dagger a + g\left[(a^\dagger)^2 + a^2\right], \tag{3.33}$$

which corresponds to the $k = 2$ case of the kth-order harmonic generation Hamiltonian (3.1). In the Bargmann-Hilbert space, the time-independent Schrödinger equation reads

$$4gz\frac{d^2}{dz^2}\psi(z) + (2\omega z + 8gq)\frac{d}{dz}\psi(z) + \left[gz + 2\omega\left(q - \frac{1}{4}\right) - E\right]\psi(z) = 0, \tag{3.34}$$

where $q = \frac{1}{4}, \frac{3}{4}$ are the Bargmann index of $su(1,1)$. This equation is the $k = 2$ special case of the kth order differential equation (3.14).

With the substitution

$$\psi(z) = e^{-\frac{\omega}{4g}(1-\Omega)z}\varphi(z), \qquad \Omega = \sqrt{1 - \frac{4g^2}{\omega^2}}, \tag{3.35}$$

where $\left|\frac{2g}{\omega}\right| < 1$, it follows,

$$\left\{4gz\frac{d^2}{dz^2} + [2\omega\Omega z + 8gq]\frac{d}{dz} + 2q\omega\Omega - \frac{1}{2}\omega - E\right\}\varphi = 0. \tag{3.36}$$

This differential equation is exactly solvable, and exact solutions are polynomials in z which automatically are entire functions in the Bargmann-Hilbert space. We thus seek solution of the form,

$$\varphi(z) \;=\; \prod_{i=1}^{\mathcal{M}}(z - z_i), \qquad \mathcal{M} = 0, 1, 2, \cdots, \tag{3.37}$$

where $\varphi(z) \equiv 1$ for $\mathcal{M} = 0$, $\mathcal{M}$ is the degree of the polynomial solution and z_i are the roots of the polynomial to be determined. Following the procedure similar to that in the last section, we obtain the energy eigenvalues,

$$E = -\frac{1}{2}\omega + \left[2\mathcal{M} + 2\left(q - \frac{1}{4}\right) + \frac{1}{2}\right]\omega\Omega, \tag{3.38}$$

and the set of algebraic equations which determine the roots z_i,

$$\sum_{j\neq i}^{\mathcal{M}}\frac{2}{z_i - z_j} + \frac{\omega}{2g}\Omega + \frac{2q}{z_i} = 0, \qquad i = 1, 2, \cdots, \mathcal{M}. \tag{3.39}$$

The corresponding wavefunctions are

$$\psi(z) = e^{-\frac{\omega}{4g}(1-\Omega)z} \prod_{i=1}^{\mathcal{M}} (z - z_i). \tag{3.40}$$

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